Thin-Walled Curved Beam Theory Based on Centroid-Shear Center Formulation

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To overcome the drawback of currently available curved beam theories having non-symmetric thin-walled cross sections, a curved beam theory based on centroid-shear center formulation is presented for the spatially coupled free vibration and elastic analysis For this, the displacement field is expressed by introducing displacement parameters defined at the centroid and shear center axes, respectively Next the elastic strain and kinetic energies considering the thickness-curvature effect and the rotary inertia of curved beam are rigorously derived by degenerating the energies of the elastic continuum to those of curved beam And then the equilibrium equations and the boundary conditions are consistently derived for curved beams having non-symmetric thin-walled cross section It is emphasized that for curved beams with L- or T-shaped sections, this thin-walled curved beam theory can be easily reduced to the solid beam theory by simply putting the sectional properties associated with warping to zero In order to illustrate the validity and the accuracy of this study, FE solutions using the Hermitian curved beam elements are presented and compared with the results by previous research and ABAQUS's shell elements

Key Words : Thin-Walled Curved Beam, Free Vibration Analysis, Elastic Analysis, Warping

1. Introduction

Curved beam structures have been used in many mechanical, aerospace and civil engineering applications such as spring design, curved wires in missile-guidance floated gyroscopes, curved girder bridges, brake shoes within drum brakes, tire dynamics, stiffeners in aircraft structures, and turbomachinery blades It can also be used as a simplified model of a shell structure

In general, the vibrational and elastic behavior of thin-walled curved beam structures are very complex because the axial, flexural and torsional deformations are coupled due to the curvature effects as well as non-symmetry of cross section. Investigation into the behavior of thin-walled curved members has been carried out extensively since the early researches (Vlasov, 1961, Timoshenko and Gere, 1961) and particularly monographs by Dabrowski (1968), Heins (1975) and Gjelsvik (1981) are worth remarking as useful references for curved beam theory and its applications

Up to the present, considerable researches (Lee, 2003, Raveendranath et al, 2000, Wilson and Lee, 1995, Gupta and Howson, 1994) on the free in-plane vibration of curved beam have been done considering the various parameters such as boundary conditions, shear deformation, rotary inertia, variable curvatures and variable cross sections. And the researches for the de-

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coupled free out-of-plane vibration behavior of curved beam have been performed by several authors (Chucheepsakul and Saetiew, 2002 , Piovan et al., 2000 , Cortinez and Piovan, 1999 , Howson and Jemah, 1999 ; Kawakami et al , 1995) Also Kang and Han (1998) presented the closed-form solution and a numerical solution for the decoupled out-of-plane static analysis of a curved beam with circular cross section subjected to torque by the differential quadrature method

It is well known that the thin-walled straight beam theory with non-symmetric cross section based on the centroid-shear center formulation is established, in which its axial, flexural and warping-torsional deformations are decoupled Hence the warping-free theory for straight beam with non-symmetric thin-walled section is easily obtained from the thin-walled beam theory by simply putting the warping moment of inertia to zero.

On the other hand, for the elastic and stability theories of curved beams based on the centroid-shear center formulation, most of previous researches (Kang and Yoo, 1994 , Yang and Kuo, 1987, 1986) have been restricted to those with doubly symmetric thin-walled cross sections Furthermore it has been reported by Gendy and Saleeb (1992) that the curved beam theory based on the centroid-shear center formulation is valid only for a cross section having doubly symmetry or one axis of symmetry which lies in the plane of beam curvature, otherwise, coupling terms still exist For this reason, it appears that most of thin-walled curved beam theories with non-symmetric cross sections have been developed based on displacement parameters which are all defined at the centroid axis (Kim et al., 2002, 2000a, b , Hu et al , 1999 , Gendy and Saleeb, 1994, 1992 , Saleeb and Gendy, 1991 ; Kim et al., 2002) presented analytical and numerical solutions on a spatial free vibration of thin-walled curved beam, as a separated curved structure, with non-symmetric section neglecting shear deformation effects and Gendy and Saleeb (1994) presented an effective formulation on spatial free vibration of arbitrary thin-walled curved beam by including the shear deformation and rotary inertia. However, they partially considered the effect of thickness-curvature and shear deformation

It is important to note that these centroid formulations for the vibration and elastic analysis of thin-walled curved beam with L- or T-shaped cross sections have a drawback to evaluate the several sectional properties associated with warping additionally because the warping function of cross section at the centroid does not become zero To the best of my knowledge, Tong and Xu's study (2002) was only the recent attempt reported on the curved beam theory with non-symmetric cross section based on the centroid-shear center formulation in the literature. However they did not consider the thickness-curvature effect which made the difference become larger in curved beam with large subtended angle and small radius and was restricted to only the elastic analysis of curved beam

The main purpose of this paper is to present a curved beam theory with non-symmetric cross section based on centroid-shear center formulation, in which the axial and flexural displacements are defined at the centroid and the lateral and warping-torsional displacements at the shear center, respectively Particularly for curved beams with L- or T-shaped sections, this thin-walled curved beam theory can be reduced easily to the theory neglecting the restrained warping torsion by simply putting the sectional properties associated with warping defined at the shear center to zero Also for the curved beam with non-symmetric closed sections, this beam theory may be reduced naturally to that with neglecting warping deformation because the values of sectional properties associated with warping at the shear center become extremely large. The important points presented are summarized as follows

(1) The displacement field for non-symmetric thin-walled curved beams with constant curvature is introduced, in which the axial displacement and two flexural rotations are defined at the centroid and the torsional rotation including the normalized warping function and two lateral displacements are defined at the shear center, respectively

(2) Next force-deformation relations due to the normal stress considering the thickness-curvature effect are accurately derived at the general coordinates.

(3) And then the elastic strain and kinetic energies based on the centroid-shear center formulation are newly derived for the free vibration and elastic analysis of non-symmetric curved beams having thin-walled open and closed cross sections, respectively.

(4) In addition, FE procedure using the Hermitian curved beam elements is presented for the analysis of non-symmetric curved beams. Finally to demonstrate the validity of the proposed study, numerical solutions are presented and compared with the results by available references and ABAQUS's shell elements.

2. Curved Beam Theory Based on the Centroid-Shear Center Formulation

To degenerate a spatially coupled free vibration and elastic theories for the continuum to those for the thin-walled curved beams, the following assumptions are adopted in this paper.

(1) The thin-walled curved beams are linearly elastic and prismatic.

(2) The cross section is rigid with respect to in-plane deformation except for warping deformation.

(3) The axis of curvature does not necessarily coincide with one of the principal axes.

2.1 Kinematics

In this study, two curvilinear coordinate systems are adopted to derive a general theory for free vibration and elastic analysis of thin-walled curved beams consistently. Fig. 1 shows the first coordinate system (x_1, x_2, x_3), in which the x_1 axis coincides with the curved centroid axis having the radius of curvature R but x_2, x_3 axes are not necessarily principal inertia axes. While the second coordinate system (x_1^s, x_2^s, x_3^s) is constituted by the shear center axis and two orthogonal axes running parallel with the direction of x_2, x_3 axes (see Fig. 2). Also x_2^p and x_3^p are principal inertia axes defined at the centroid. Then transformation equations between two coordinates systems may be expressed by

$$x_1 = x_1^s \tag{1a}$$

$$x_2 = x_2^s + e_2 = x_2^p \cos\gamma - x_3^p \sin\gamma \tag{1b}$$

$$x_3 = x_3^s + e_3 = x_2^p \sin\gamma + x_3^p \cos\gamma \tag{1c}$$

where (e_2, e_3) denotes the position vector of the

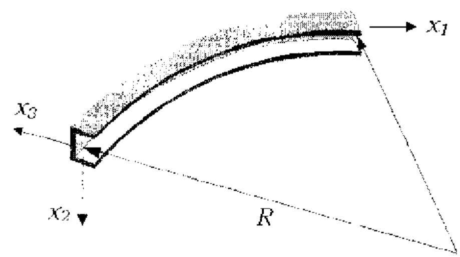
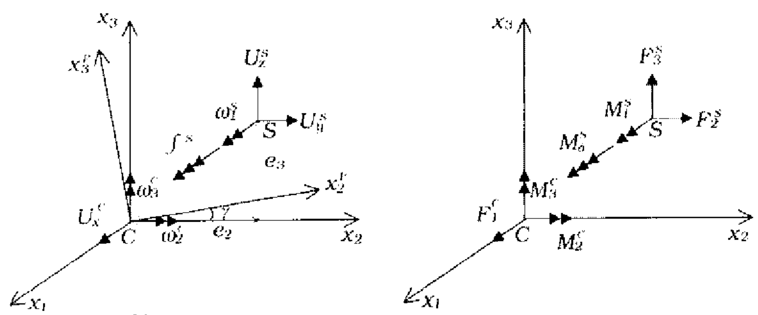


Fig. 1 A curvilinear coordinate system for non-symmetric thin-walled curved beam



S : Shear center

C : Centroid

(a) Displacement parameters (b) Stress resultants

Fig. 2 Two coordinate systems, displacement parameters and stress resultants shear center and γ is the angle between x_2 and the x_2^p axis

To introduce the displacement field for the non-symmetric thin-walled cross section, seven displacement parameters and stress resultants are used as shown in Figs 2(a) and 2(b), respectively Assuming that the cross section is rigid with respect to in-plane deformation, the displacement field can be written as follows

$$U_1 = U_x + x_3\omega_2 - x_2\omega_3 + f\phi(x_2, x_3) \tag{2a}$$

$$U_2 = U_y - \theta(x_3 - e_3) \tag{2b}$$

$$U_3 = U_z + \theta(x_2 - e_2) \tag{2c}$$

where U_x, ω_2, ω_3=the rigid body translation and two rotations with respect to x_1, x_2, x_3 axes ; θ, U_y, U_z=the rigid body rotation and two translations with respect to x_1^s, x_2^s, x_3^s axes , f, ϕ=the displacement parameter measuring warping deformation and the normalized warping function defined at the shear center, respectively For later use, sectional properties with respect to the centroid-shear center are defined as

$$I_2 = \int_A x_3^2\, dA,\ I_3 = \int_A x_2^2\, dA,\ I_{23} = \int_A x_2 x_3\, dA$$
$$I_\phi = \int_A \phi^2\, dA,\ I_{\phi 2} = \int_A \phi x_3\, dA,\ I_{\phi 3} = \int_A \phi x_2\, dA \tag{3a-l}$$
$$I_{222} = \int_A x_3^3\, dA,\ I_{223} = \int_A x_2 x_3^2\, dA,\ I_{233} = \int_A x_2^2 x_3\, dA$$
$$I_{\phi 22} = \int_A \phi x_3^2\, dA,\ I_{\phi 23} = \int_A \phi x_2 x_3\, dA,\ I_{\phi\phi 2} = \int_A \phi^2 x_3\, dA$$

where A, I_2, I_3 and I_{23}=the cross sectional area, the second moments of inertia and the product moment of inertia about x_2 and x_3 axes, respectively I_ϕ=the warping moment of inertia It should be noticed that $I_{\phi 2}$, $I_{\phi 3}$ are always equal to zero and I_{222}, I_{223}, I_{233}, $I_{\phi 22}$, $I_{\phi 23}$, $I_{\phi\phi 2}$ denote the sectional properties to consider the thickness-curvature effect which makes the difference become larger in curved beam with large subtended angle and small radius

2.2 Principle of virtual work

With the assumption of the rigid in-plane deformation, stress resultants with respect to the centroid-shear center axes are defined as follows

$$F_1 = \int_A \tau_{11}\, dA,\ F_2 = \int_A \tau_{12}\, dx,\ F_3 = \int_A \tau_{13}\, dA$$
$$M_1 = \int_A [\tau_{13}(x_2 - e_2) - \tau_{12}(x_3 - e_3)]\, dA$$
$$M_2 = \int_A \tau_{11} x_3\, dA,\ M_3 = -\int_A \tau_{11} x_2\, dA,\ M_\phi = \int_A \tau_{11}\phi\, dA \tag{4a-h}$$
$$M_R = \int_A \left[\tau_{12}\phi_{,2} + \tau_{13}\left(\phi_{,3} - \frac{\phi}{R + x_3}\right)\right]\frac{R + x_3}{R}\, dA$$

where F_1=the axial force acting at the centroid , F_2 and F_3=the shear forces acting at the shear center , M_1=the total twisting moment with respect to the shear center axis , M_2 and M_3=the bending moments with respect to x_2 and x_3 axes, respectively M_R and M_ϕ=the restrained (non-uniform) torsional moment and the bimoment about the shear center axis, respectively

The principle of virtual work for the general continuum vibrating harmonically is expressed as

$$\int_V \tau_{ij}\delta e_{ij}\, dV - \omega^2\int_V \rho U_i \delta U_i\, dV = \int_S T_i \delta U_i\, dS \tag{5}$$

where e_{ij}=the conventional linear strain due to U_i , ρ=the density , ω=the circular frequency , T_i=the surface force The first term denotes the conventional internal virtual work giving the elastic strain energy and the second term represents the kinetic energy In case of the thin-walled circular beam, Eq (5) may be transformed to the principle of the total potential energy Π as follows

$$\Pi = \Pi_E - \Pi_M - \Pi_{ext} \tag{6}$$

where the detailed expressions for each term of Π are

$$\Pi_E = \frac{1}{2}\int_o^l \int_A [\tau_{11}e_{11} + 2\tau_{12}e_{12} + 2\tau_{13}e_{13}]\frac{R + x_3}{R}\, dA\, dx_1 \tag{7a}$$

$$\Pi_M = \frac{1}{2}\rho\omega^2\int_o^l \int_A [U_1^2 + U_2^2 + U_3^2]\frac{R + x_3}{R}\, dA\, dx_1 \tag{7b}$$

$$\Pi_{ext} = \frac{1}{2}\mathbf{U}_e^T\mathbf{F}_e \tag{7c}$$

where $\mathbf{U}_e$, $\mathbf{F}_e$=the nodal displacement and nodal force vectors, respectively

On the other hand, strain-displacement relations due to the fiist order displacements are expressed as follows

$$
\begin{aligned}
e_{11}&=\left(U_{1,1}+\frac{U_3}{R}\right)\frac{R}{R+x_3}\\
&=\left[\left(U_x'+\frac{U_z}{R}-\frac{e_2}{R}\theta\right)\right.\\
&\quad\left.-x_2\left(-\frac{\theta}{R}+\omega_3'\right)+x_3\omega_2'+\phi f'\right]\frac{R}{R+x_3}
\end{aligned}
\tag{8a}
$$

$$
\begin{aligned}
2e_{12}&=\frac{U_{2,1}R}{R+x_3}+U_{1,2}\\
&=[U_y'-\theta'(x_3-e_3)]\frac{R}{R+x_3}-\omega_3+f\phi_{,2}
\end{aligned}
\tag{8b}
$$

$$
\begin{aligned}
2e_{13}&=\left(U_{3,1}-\frac{U_1}{R}\right)\frac{R}{R+x_3}+U_{1,3}\\
&=\left[-\frac{U_x}{R}+U_z'+\theta'(x_2-e_2)-\frac{x_3}{R}\omega_2\right.\\
&\quad\left.+\frac{x_2}{R}\omega_3-\frac{f}{R}\phi\right]\frac{R}{R+x_3}+\omega_2+f\phi_{,3}
\end{aligned}
\tag{8c}
$$

For thin-walled circular beams subjected to distributed loadings, by substituting linear strains (8a-c) into Eq (7a) and integrating over the cross sectional area, Eq (7a) is reduced to the following equations

$$
\begin{aligned}
\Pi_E=\frac{1}{2}\int_o^l&\left[F_1\left(U_x'+\frac{U_z}{R}-\frac{e_2}{R}\theta\right)+M_2\omega_2'+M_3\left(-\frac{\theta}{R}+\omega_3'\right)+M_\phi f'\right.\\
&+F_2\left(U_y'-\omega_3-\frac{e_3}{R}\omega_3+\frac{e_3^2}{R}f\right)\\
&+F_3\left(-\frac{U_x}{R}+U_z'+\omega_2+\frac{e_2}{R}\omega_3-\frac{e_2e_3}{R}f\right)\\
&\left.+(M_1-M_R)\left(\theta'+\frac{\omega_3}{R}-\frac{e_3}{R}f\right)+M_R\left(\theta'+\frac{\omega_3}{R}+f-\frac{e_3}{R}f\right)\right]dx_1
\end{aligned}
\tag{9}
$$

And Eq (7c) can be expressed as

$$
\begin{aligned}
\Pi_{ext}=\int_o^l&[p_1U_x+p_2U_y+p_3U_z\\
&+m_1\omega_1+m_2\omega_2+m_3\omega_3+m_\phi f]dx_1
\end{aligned}
\tag{10}
$$

wheie p_1, p_2, p_3 are the distributed forces in the direction of x_1, x_2, x_3 axes and m_1, m_2, m_3, m_ϕ denote distributed moments

Now by invoking the stationary condition of the total potential energy, equilibrium equations and boundary conditions are obtained as

$$F_1'+\frac{F_3}{R}=-p_1 \tag{11a}$$

$$F_2'=-p_2 \tag{11b}$$

$$-\frac{F_1}{R}+F_3'=-p_3 \tag{11c}$$

$$\frac{e_2}{R}F_1+M_1'+\frac{M_3}{R}=-m_1 \tag{11d}$$

$$-F_3+M_2'=-m_2 \tag{11e}$$

$$F_2+\frac{e_3}{R}F_2-\frac{e_2}{R}F_3-\frac{M_1}{R}+M_3'=-m_3 \tag{11f}$$

$$
\begin{aligned}
&-\frac{e_3^2}{R}F_2+\frac{e_2e_3}{R}F_3+\frac{e_3}{R}M_1-\frac{e_3}{R}M_R\\
&-M_R+M_\phi'=-m_\phi
\end{aligned}
\tag{11g}
$$

and

$$\delta U_x(o)=\delta U_x^p \text{ or } F_1(o)=-F_1^p \tag{12a}$$

$$\delta U_x(l)=\delta U_x^q \text{ oi } F_1(l)=F_1^q \tag{12b}$$

$$\delta U_y(o)=\delta U_y^p \text{ or } F_2(o)=-F_2^p \tag{12c}$$

$$\delta U_y(l)=\delta U_y^q \text{ or } F_2(l)=F_2^q \tag{12d}$$

$$\delta U_z(o)=\delta U_z^p \text{ or } F_3(o)=-F_3^p \tag{12e}$$

$$\delta U_z(l)=\delta U_z^q \text{ or } F_3(l)=F_3^q \tag{12f}$$

$$\delta\theta(o)=\delta\theta^p \text{ or } M_1(o)=-M_1^p \tag{12g}$$

$$\delta\theta(l)=\delta\theta^q \text{ or } M_1(l)=M_1^q \tag{12h}$$

$$\delta\omega_2(o)=\delta\omega_2^p \text{ or } M_2(o)=-M_2^p \tag{12i}$$

$$\delta\omega_2(l)=\delta\omega_2^q \text{ or } M_2(l)=M_2^q \tag{12j}$$

$$\delta\omega_3(o)=\delta\omega_3^p \text{ or } M_3(o)=-M_3^p \tag{12k}$$

$$\delta\omega_3(l)=\delta\omega_3^q \text{ or } M_3(l)=M_3^q \tag{12l}$$

$$\delta f(o)=\delta f^p \text{ or } M_\phi(o)=-M_\phi^p \tag{12m}$$

$$\delta f(l)=\delta f^q \text{ or } M_\phi(l)=M_\phi^q \tag{12n}$$

2.3 Elastic strain and kinetic energies of thin-walled curved beam

Now force-deformation relations due to the normal stress are derived In this study, the sheai deformation effects due to both the shear forces and the restrained warping torsion are neglected Therefore, the shear rigidity constraints in Eq (9) are as follows

$$U_y'-\omega_3-\frac{e_3}{R}\omega_3+\frac{e_3^2}{R}f=0 \tag{13a}$$

$$-\frac{U_x}{R}+U_z'+\omega_2+\frac{e_2}{R}\omega_3-\frac{e_2e_3}{R}f=0 \quad (13b)$$

$$\theta'+\frac{\omega_3}{R}+f-\frac{e_3}{R}f=0 \quad (13c)$$

Form Eq (13a-c), the rotational displacements ω_2, ω_3, and the warping parameter f may be rewritten with respect to U_x, U_y, U_z, θ as follows

$$\omega_2=\frac{U_x}{R}-\frac{e_2}{R}U_y'-U_z'-\frac{e_2e_3}{R}\theta' \quad (14a)$$

$$\omega_3=\frac{R-e_3}{R}U_y'-\frac{e_3^2}{R}\theta' \quad (14b)$$

$$f=-\frac{U_y'}{R}-\frac{R+e_3}{R}\theta' \quad (14c)$$

Accordingly we can rewrite the displacement field U_1 in Eq (2a) using Eqs (14a-c)

$$\begin{aligned}U_1=U_x&-x_2\left\{\left(1-\frac{e_3}{R}\right)U_y'-\frac{e_3^2}{R}\theta'\right\}\\&-x_3\left\{-\frac{U_x}{R}+\frac{e_2}{R}U_y'+U_z'+\frac{e_2e_3}{R}\theta'\right\}\\&-\left\{\frac{U_y'}{R}+\left(1+\frac{e_3}{R}\right)\theta'\right\}\phi\end{aligned} \quad (15)$$

Also the normal strain e_{11} can be obtained by substituting Eqs. (14a-c) into Eq (8a) And then by substituting Eq. (8a) into Eqs (4a), (4e), (4f), (4g) and integrating over the cross section, the following force-deformation relations due to the normal stress are obtained

$$\begin{Bmatrix}F_1\\M_2\\M_3\\M_\phi\end{Bmatrix}=E\begin{bmatrix}A+\frac{\hat{I}_2}{R^2} & -\frac{\hat{I}_2}{R} & \frac{\hat{I}_{23}}{R} & \frac{I_{\phi22}}{R^2}\\ -\frac{\hat{I}_2}{R} & \hat{I}_2 & -\hat{I}_{23} & -\frac{I_{\phi22}}{R}\\ \frac{\hat{I}_{23}}{R} & -\hat{I}_{23} & \hat{I}_3 & \frac{I_{\phi23}}{R}\\ \frac{I_{\phi22}}{R^2} & -\frac{I_{\phi22}}{R} & \frac{I_{\phi23}}{R} & \hat{I}_\phi\end{bmatrix}\begin{Bmatrix}U_x'+\frac{U_z}{R}-\frac{e_2}{R}\theta\\ \frac{U_x'}{R}-\frac{e_2}{R}U_y''-U_z''-\frac{e_2e_3}{R}\theta''\\ \frac{R-e_3}{R}U_y''-\frac{e_3^2}{R}\theta''-\frac{\theta}{R}\\ -\frac{U_y''}{R}-\frac{R+e_3}{R}\theta''\end{Bmatrix} \quad (16a\text{-}d)$$

where E=the Young's modulus and

$$\hat{I}_2=I_2-\frac{I_{222}}{R},\ \hat{I}_3=I_3-\frac{I_{233}}{R} \quad (17a,b)$$

$$\hat{I}_{23}=I_{23}-\frac{I_{223}}{R},\ \hat{I}_\phi=I_\phi-\frac{I_{\phi\phi2}}{R} \quad (17c,d)$$

And the St Venant torsional moment is expressed as

$$M_{st}=GJ\left(\frac{U_y'}{R}+\frac{R+e_3}{R}\theta'\right) \quad (18)$$

where G=the shear modulus and J=the torsional constant In evaluating Eqs (16a-d), $I_{\phi2}$ and $I_{\phi3}$ vanish and the following approximation is used

$$\frac{R}{R+x_3}\cong 1-\frac{x_3}{R}+\left(\frac{x_3}{R}\right)^2 \quad (19)$$

Consequently substitution of force-deformation relations (16a-d) and (18) into Eq (9) leads to the elastic strain energy of the thin-walled curved beam with non-symmetric cross section.

$$\begin{aligned}\Pi_E=\frac{1}{2}\int_0^l\Big[&EA\left(U_x'+\frac{U_z}{R}-\frac{e_2}{R}\theta\right)^2\\&+E\hat{I}_2\left(\frac{e_2}{R}U_y''+U_z''+\frac{U_z}{R^2}+\frac{e_2e_3}{R}\theta''-\frac{e_2}{R^2}\theta\right)^2\\&+E\hat{I}_3\left(\frac{R-e_3}{R}U_y''-\frac{e_3^2}{R}\theta''-\frac{\theta}{R}\right)^2\\&+E\hat{I}_\phi\left(\frac{U_y''}{R}+\frac{R+e_3}{R}\theta''\right)^2+GJ\left(\frac{U_y'}{R}+\frac{R+e_3}{R}\theta'\right)^2\\&+2E\hat{I}_{23}\left(\frac{e_2}{R}U_y''+U_z''+\frac{U_z}{R^2}+\frac{e_2e_3}{R}\theta''-\frac{e_2}{R^2}\theta\right)\left(\frac{R-e_3}{R}U_y''-\frac{e_3^2}{R}\theta''-\frac{\theta}{R}\right)\\&-2\frac{EI_{\phi22}}{R}\left(\frac{U_y''}{R}+\frac{R+e_3}{R}\theta''\right)\left(\frac{e_2}{R}U_y''+\frac{U_z}{R^2}+U_z''+\frac{e_2e_3}{R}\theta''-\frac{e_2}{R^2}\theta\right)\\&-2\frac{EI_{\phi23}}{R}\left(\frac{U_y''}{R}+\frac{R+e_3}{R}\theta''\right)\left(\frac{R-e_3}{R}U_y''-\frac{e_3^2}{R}\theta''-\frac{\theta}{R}\right)\Big]dx_1\end{aligned} \quad (20)$$

Now by eliminating F_2, F_3, M_R from Eqs (11a-g), equilibrium equations of curved beams become

$$F_1'+\frac{M_2'}{R}=-p_1-\frac{m_2}{R} \quad (21a)$$

$$-\frac{M_1'}{R}-\frac{e_2}{R}M_2''+M_3''=\frac{R+e_3}{R}p_2+\frac{e_2}{R}m_2'-m_3' \quad (21b)$$

$$-\frac{F_1}{R}+M_2''=-p_3-m_2' \quad (21c)$$

$$\frac{e_2}{R}F_1+\frac{e_2e_3}{R}M_2''+\frac{M_3}{R}+\frac{R+e_3}{R}M_{st}'+M_\phi'' \\ =-\frac{e_3^2}{R}p_2-m_1-\frac{e_2e_3}{R}m_2'-m_\phi' \tag{21d}$$

And substitution of Eqs (16a-d) and (18) into Eqs (21a-d) results in

$$EA\left(U_x''+\frac{U_z'}{R}-\frac{e_2}{R}\theta'\right)=-p_1-\frac{m_2}{R} \tag{22a}$$

$$\begin{aligned}&\frac{e_2}{R}E\hat{I}_2\left(\frac{e_2}{R}U_y''''+U_z''''+\frac{U_z''}{R^2}+\frac{e_2e_3}{R}\theta''''-\frac{e_2}{R^2}\theta''\right)\\&+E\hat{I}_3\left(\frac{R-e_3}{R}U_y''''-\frac{e_3^2}{R}\theta''''-\frac{\theta''}{R}\right)\\&+E\hat{I}_{23}\left(\frac{e_2(2R-e_3)}{R^2}U_y''''+U_z''''+\frac{U_z''}{R^2}\right.\\&\left.+\frac{e_2e_3(R-e_3)}{R^2}\theta''''-\frac{2e_2}{R^2}\theta''\right)\\&-\frac{GJ}{R}\left(\frac{U_y''}{R}+\frac{R+e_3}{R}\theta''\right)\\&+\frac{E\hat{I}_\phi}{R}\left(\frac{U_y''}{R}+\frac{R+e_3}{R}\theta''\right)\\&=\frac{R+e_3}{R}p_2+\frac{e_2}{R}m_2'-m_3'\end{aligned} \tag{22b}$$

$$\begin{aligned}&E\hat{I}_2\left\{-\frac{e_2}{R}U_y''''-\frac{e_2}{R^3}U_y''-U_z''''-2\frac{U_z''}{R^2}-\frac{U_z}{R^4}\right.\\&\left.-\frac{e_2e_3}{R}\theta''''+\frac{e_2}{R^2}\left(1-\frac{e_3}{R}\right)\theta''+\frac{e_2}{R^4}\theta\right\}\\&-E\hat{I}_{23}\left(\frac{R-e_3}{R}U_y''''-\frac{R-e_3}{R^3}U_y''+\frac{e_3^2}{R}\theta''''+\frac{\theta''}{R}+\frac{\theta}{R^3}\right)\\&-\frac{EA}{R}\left(U_x'+\frac{U_z}{R}-\frac{e_2}{R}\theta\right)=-p_3-m_2'\end{aligned} \tag{22c}$$

$$\begin{aligned}&\frac{e_2}{R}EA\left(U_x'+\frac{U_z}{R}-\frac{e_2}{R}\theta\right)+\frac{e_2}{R^2}E\hat{I}_2\left(-e_2e_3U_y''''-\frac{e_2}{R}U_y''\right.\\&\left.-Re_3U_z''''+\frac{R-e_3}{R}U_z''+\frac{U_z}{R^2}-e_2e_3^2\theta''''+\frac{2e_2e_3}{R}\theta''-\frac{e_2}{R^2}\theta\right)\\&+E\hat{I}_3\left(\frac{R-e_3}{R^2}U_y''-\frac{e_3}{R^2}\theta''-\frac{\theta}{R^2}\right)\\&+E\hat{I}_{23}\left\{-\frac{e_2e_3(R-e_3)}{R^2}U_y''''+\frac{e_2(2R-e_3)}{R^3}U_y''\right.\\&\left.+\frac{U_z''}{R}+\frac{U_z}{R^3}+\frac{e_2e_3^3}{R^2}\theta''''+\frac{e_2e_3}{R}\left(2-\frac{e_3}{R}\right)\theta''-\frac{2e_2}{R^3}\theta\right\}\\&+GJ\left\{\frac{R+e_3}{R^2}U_y''+\frac{(R+e_3)^2}{R^2}\theta''\right\}\\&-E\hat{I}_\phi\left(\frac{U_y''''}{R}+\frac{R+e_3}{R}\theta''''\right)=-\frac{e_3^2}{R}p_2-m_1-\frac{e_2e_3}{R}m_2'-m_\phi'\end{aligned} \tag{22d}$$

Also we can obtain the kinetic energy Π_M by substituting the displacement field in Eqs (2b,c) and (15) into Eq. (7b) and as follows

$$\begin{aligned}\Pi_M=\frac{1}{2}\rho\omega^2\int_0^l&\left[A\{U_x^2+U_y^2+U_z^2+\theta^2(e_2^2+e_3^2)+2\theta(e_3U_y-e_2U_z)\}\right.\\&+\tilde{I}_3\left(\frac{R-e_3}{R}U_y'-\frac{e_3^2}{R}\theta'\right)^2+\tilde{I}_2\left(U_z'-\frac{U_x}{R}+\frac{e_2}{R}U_y'+\frac{e_2e_3}{R}\theta'\right)^2\\&+\tilde{I}_\phi\left(\frac{R+e_3}{R}\theta'+\frac{U_y'}{R}\right)^2-2\frac{I_{23}}{R}U_x\left(\frac{R-e_3}{R}U_y'-\frac{e_3^2}{R}\theta'\right)\\&-2\frac{I_2}{R}U_x\left(U_z'-\frac{U_x}{R}+\frac{e_2}{R}U_y'+\frac{e_2e_3}{R}\theta'\right)\\&+2\tilde{I}_{23}\left(\frac{R-e_3}{R}U_y'-\frac{e_3^2}{R}\theta'\right)\left(U_z'-\frac{U_x}{R}+\frac{e_2}{R}U_y'+\frac{e_2e_3}{R}\theta'\right)\\&+2\frac{I_{\phi23}}{R}\left(\frac{R-e_3}{R}U_y'-\frac{e_3^2}{R}\theta'\right)\left(\frac{R+e_3}{R}\theta'+\frac{U_y'}{R}\right)\\&+\left(\tilde{I}_o-2\frac{I_2e_3}{R}-2\frac{I_{23}e_2}{R}\right)\theta^2+2\frac{\theta}{R}(I_{23}U_z-I_2U_y)\\&\left.+2\frac{I_{\phi22}}{R}\left(U_z'-\frac{U_x}{R}+\frac{e_2}{R}U_y'+\frac{e_2e_3}{R}\theta'\right)\left(\frac{R+e_3}{R}\theta'+\frac{U_y'}{R}\right)\right]dx_1\end{aligned} \tag{23}$$

where

$$\tilde{I}_2=I_2+\frac{I_{222}}{R},\ \tilde{I}_3=I_3+\frac{I_{233}}{R},\ \tilde{I}_{23}=I_{23}+\frac{I_{223}}{R} \\ \tilde{I}_\phi=I_\phi+\frac{I_{\phi\phi2}}{R},\ \tilde{I}_o=I_2+I_3+\frac{I_{222}+I_{233}}{R} \tag{24}$$

As mentioned previously, for the curved beams with cross sections neglecting the warping function at the shear center such as L- or T-shaped cross sections, the sectional properties (i.e., $\hat{I}_\phi$, $I_{\phi22}$, $I_{\phi23}$, $\tilde{I}_\phi$) in Eqs. (20) and (23) associated with warping become zero obviously Also for curved beams with non-symmetric closed sections, these properties have the extremely large values so that those can be interpreted as penalty numbers in the strain energy of curved beams Resultantly this means that strain and kinetic energy terms related to warping should vanish in the centroid-shear center formulation for the curved beams with L- or T-shaped cross sections or closed sections

Based on these reasons, for the spatially coupled vibration and elastic analysis of curved beams with thin-walled open cross sections having the warping function vanishing at the shear center or with thin-walled closed cross sections, the elastic strain and kinetic energy expression can be easily simplified to Eqs (25) and (26), respectively

$$\begin{aligned}\Pi_E^* = \frac{1}{2}\int_0^l \Big[& EA\left(U_x' + \frac{U_z}{R} - \frac{e_2}{R}\theta\right)^2 \\ & + E\hat{I}_2\left(\frac{e_2}{R}U_y'' + \frac{U_z}{R^2} + U_z'' + \frac{e_2 e_3}{R}\theta'' - \frac{e_2}{R^2}\theta\right)^2 \\ & + E\hat{I}_3\left(\frac{R-e_3}{R}U_y'' - \frac{e_3^2}{R}\theta'' - \frac{\theta}{R}\right)^2 + GJ\left(\frac{U_y'}{R} + \frac{R+e_3}{R}\theta'\right)^2 \\ & + 2E\hat{I}_{23}\left(\frac{e_2}{R}U_y'' + U_z'' + \frac{U_z}{R^2} + \frac{e_2 e_3}{R}\theta'' - \frac{e_2}{R^2}\theta\right) \\ & \quad \left(\frac{R-e_3}{R}U_y'' - \frac{e_3^2}{R}\theta'' - \frac{\theta}{R}\right)\Big] dx_1\end{aligned} \tag{25}$$

and

$$\begin{aligned}\Pi_M^* = \frac{1}{2}\rho\omega^2\int_0^l \Big[& A\{U_x^2 + U_y^2 + U_z^2 + \theta^2(e_2^2 + e_3^2) + 2\theta(e_3 U_y - e_2 U_z)\} \\ & + \tilde{I}_3\left(\frac{R-e_3}{R}U_y' - \frac{e_3^2}{R}\theta'\right)^2 + \tilde{I}_2\left(U_z' - \frac{U_x}{R} + \frac{e_2}{R}U_y' + \frac{e_2 e_3}{R}\theta'\right)^2 \\ & - 2\frac{I_{23}}{R}U_x\left(\frac{R-e_3}{R}U_y' - \frac{e_3^2}{R}\theta'\right) + 2\frac{\theta}{R}(I_{23}U_z - I_2 U_y) \\ & - 2\frac{I_2}{R}U_x\left(U_z' - \frac{U_x}{R} + \frac{e_2}{R}U_y' + \frac{e_2 e_3}{R}\theta'\right) \\ & + 2\tilde{I}_{23}\left(\frac{R-e_3}{R}U_y' - \frac{e_3^2}{R}\theta'\right)\left(U_z' - \frac{U_x}{R} + \frac{e_2}{R}U_y' + \frac{e_2 e_3}{R}\theta'\right) \\ & + \left(\tilde{I}_o - 2\frac{I_2 e_3}{R} - 2\frac{I_{23}e_2}{R}\right)\theta^2\Big] dx_1\end{aligned} \tag{26}$$

On the other hand, Kim et al (2002) used following elastic strain and kinetic energies for the spatially coupled free vibration analysis of curved beam with non-symmetric cross section including the L- or T-shaped sections based on the centroid formulation in which the seven displacement parameters are defined at the centroid

$$\begin{aligned}\Pi_E^C = \frac{1}{2}\int_0^l \Big[& EA\left(U_x^{c'} + \frac{U_z^c}{R}\right)^2 + E\hat{I}_2\left(U_z^{c''} + \frac{U_z^c}{R^2}\right)^2 \\ & + E\hat{I}_3\left(U_y^{c''} - \frac{\theta^c}{R}\right)^2 + GJ\left(\frac{U_y^{c'}}{R} + \theta^{c'}\right)^2 \\ & + 2E\hat{I}_{23}\left(U_z^{c''} + \frac{U_z^c}{R^2}\right)\left(U_y^{c''} - \frac{\theta^c}{R}\right) + E\hat{I}_\phi^c\left(\frac{U_y^{c''}}{R} + \theta^{c''}\right)^2 \\ & + 2E\hat{I}_{\phi 2}^c\left(U_z^{c''} + \frac{U_z^c}{R^2}\right)\left(\frac{U_y^{c''}}{R} + \theta^{c''}\right) \\ & + 2E\hat{I}_{\phi 3}^c\left(U_y^{c''} - \frac{\theta^c}{R}\right)\left(\frac{U_y^{c''}}{R} + \theta^{c''}\right)\Big] dx_1\end{aligned} \tag{27}$$

where

$$\hat{I}_\phi^c = I_\phi^c - \frac{I_{\phi\phi 2}^c}{R},\ \hat{I}_{\phi 2}^c = I_{\phi 2}^c - \frac{I_{\phi 22}^c}{R},\ \hat{I}_{\phi 3}^c = I_{\phi 3}^c - \frac{I_{\phi 23}^c}{R} \tag{28a-c}$$

and

$$\begin{aligned}\Pi_M^C = \frac{1}{2}\rho\omega^2\int_0^l \Big[& A(U_x^{c2} + U_y^{c2} + U_z^{c2}) + \tilde{I}_2\left(U_z^{c'} - \frac{U_x}{R}\right)^2 \\ & - 2\frac{I_2}{R}\left\{U_y^c\theta^c + U_x\left(U_z^c - \frac{U_x}{R}\right)\right\} \\ & + \tilde{I}_3 U_y^{c'2} + 2I_{23}\left(U_y^{c'}U_z^c - \frac{2}{R}U_x U_y^c + \frac{1}{R}U_z^c\theta^c\right) \\ & + 2\frac{I_{223}}{R}U_y^c\left(U_z^c - \frac{U_x}{R}\right)^2 + \tilde{I}_o\theta^{c2} \\ & + \tilde{I}_\phi^c\left(\frac{U_y^c}{R} + \theta^{c'}\right)^2 + 2\tilde{I}_{\phi 2}^c\left(U_z^c - \frac{U_x}{R}\right)\left(\frac{U_y^{c'}}{R} + \theta^c\right) \\ & - 2\frac{I_{\phi 2}^c}{R}U_x\left(\frac{U_y^c}{R} + \theta^c\right) + 2\tilde{I}_{\phi 3}^c U_y'\left(\frac{U_y^c}{R} + \theta^c\right)\Big] dx_1\end{aligned} \tag{29}$$

where

$$\tilde{I}_\phi^c = I_\phi^c + \frac{I_{\phi\phi 2}^c}{R},\ \tilde{I}_{\phi 2}^c = I_{\phi 2}^c + \frac{I_{\phi 22}^c}{R},\ \tilde{I}_{\phi 3}^c = I_{\phi 3}^c + \frac{I_{\phi 23}^c}{R} \tag{30a-c}$$

The transformation equations between the sectional properties associated with warping which are defined at the centroid and those at the shear center can be obtained For this, the kinematical relationship between ϕ^c and ϕ defined at the centroid and the shear center, respectively, can be expressed as

$$\phi^c = \phi + e_2 x_3 - e_3 x_2 \tag{31}$$

Then the transformation equations may be expressed as follows

$$\begin{aligned} I_\phi^c &= \int_A \phi^{c2} dA = \int_A (\phi + e_2 x_3 - e_3 x_2)^2 dA \\ &= I_\phi + e_2^2 I_2 + e_3^2 I_3 - 2e_2 e_3 I_{23} \end{aligned} \tag{32a}$$

$$\begin{aligned} I_{\phi 2}^c &= \int_A \phi^c x_3 dA = \int_A (\phi + e_2 x_3 - e_3 x_2) x_3 dA \\ &= e_2 I_2 - e_3 I_{23} \end{aligned} \tag{32b}$$

$$\begin{aligned} I_{\phi 3}^c &= \int_A \phi^c x_2 dA = \int_A (\phi + e_2 x_3 - e_3 x_2) x_2 dA \\ &= -e_3 I_3 + e_2 I_{23} \end{aligned} \tag{32c}$$

$$\begin{aligned} I_{\phi 23}^c &= \int_A \phi^c x_2 x_3 dA = \int_A (\phi + e_2 x_3 - e_3 x_2) x_2 x_3 dA \\ &= I_{\phi 23} + e_2 I_{223} - e_3 I_{233} \end{aligned} \tag{32d}$$

$$\begin{aligned} I_{\phi 22}^c &= \int_A \phi^c x_3^2 dA = \int_A (\phi + e_2 x_3 - e_3 x_2) x_3^2 dA \\ &= I_{\phi 22} + e_2 I_{222} - e_3 I_{223} \end{aligned} \tag{32e}$$

$$\begin{aligned} I_{\phi\phi 2}^c &= \int_A \phi^{c2} x_3 dA = \int_A (\phi + e_2 x_3 - e_3 x_2)^2 x_3 dA \\ &= I_{\phi\phi 2} + e_2^2 I_{222} + e_3^2 I_{233} + 2e_2 I_{\phi 22} \\ &\quad - 2e_3 I_{\phi 23} - 2e_2 e_3 I_{223} \end{aligned} \tag{32f}$$

Here it should be noticed that when Eq. (25) compares with Eqs. (27) and (26) with Eq. (29), one can find a drawback in the previous formulation (Kim et al., 2002), namely the elastic strain and kinetic energies from the centroid formulation should retain the several sectional properties related to the warping function which does not become zero at the centroid.

3. Finite Element Formulation

The Hermitian curved beam element having arbitrary thin-walled cross sections is used based on the elastic strain and kinetic energy expressions derived in the previous Section. Fig. 3 shows the nodal displacement vector of thin-walled Hermitian curved beam element including restrained warping effect. This curved beam element has two nodes and eight degrees of freedom per node. As a result, the element displacement parameters U_x, U_y, U_z, θ can be interpolated with respect to the nodal displacements, which the detailed expression is presented in Kim et al. (2002). By substituting the interpolating functions, material and cross-sectional properties into Eqs. (20), (23) and (10) and integrating along the element length, equations of motion of thin-walled curved beam element are obtained in matrix form as

$$(\mathbf{K}_e-\omega^2\mathbf{M}_e)\,\mathbf{U}_e=\mathbf{F}_e \tag{33}$$

where

$$\mathbf{U}_e=\left\langle \begin{matrix} u^p,\ v^p,\ w^p,\ \omega_1^p,\ \omega_2^p,\ \omega_3^p,\ f^p,\ g^p \\ u^q,\ v^q,\ w^q,\ \omega_1^q,\ \omega_2^q,\ \omega_3^q,\ f^q,\ g^q \end{matrix} \right\rangle \tag{34a}$$

$$\mathbf{F}_e=\left\langle \begin{matrix} F_1^p,\ F_2^p,\ F_3^p,\ M_1^p,\ M_2^p,\ M_3^p,\ M_\phi^p,\ F_m^p, \\ F_1^q,\ F_2^q,\ F_3^q,\ M_1^q,\ M_2^q,\ M_3^q,\ M_\phi^q,\ F_m^q \end{matrix} \right\rangle \tag{34b}$$

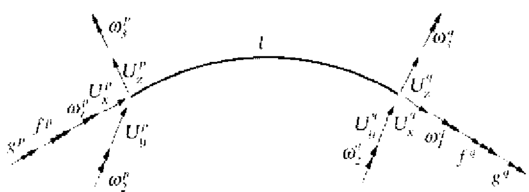


Fig. 3 Nodal displacement vector of Hermitian curved beam element

In the above equation, $\mathbf{K}_e$ is the 1616 element elastic stiffness matrix in local coordinate. In this study, stiffness matrices are evaluated using a Gauss numerical integration scheme. For Eq. (34a), it is convenient to transform the rotational and axial nodal displacement components into the nodal components including curvature effect as following

$$\tilde{\omega}_2^p=-U_z'(o)+\frac{U_x(o)}{R}=\omega_2^p+\frac{u^p}{R} \tag{35a}$$

$$\tilde{f}^p=-\theta'(o)-\frac{U_y'(o)}{R}=f^p+\frac{\omega_3^p}{R} \tag{35b}$$

$$\tilde{g}^p=U_x'(o)+\frac{U_z(o)}{R}=g^p+\frac{w^p}{R} \tag{35c}$$

For the evaluation of the element stiffness matrix corresponding to the transformed nodal displacements, the transformation between the member displacement vectors of Eqs. (34a) and the member displacement considering the effect of curvature is expressed as

$$\mathbf{U}_\zeta=\mathbf{T}_1\tilde{\mathbf{U}}_\zeta,\ \zeta=p,\ q \tag{36}$$

where

$$\mathbf{U}_p^T=\{u^p,\ v^p,\ w^p,\ \omega_1^p,\ \omega_2^p,\ \omega_3^p,\ f^p,\ g^p\} \tag{37a}$$

$$\tilde{\mathbf{U}}_p^T=\{u^p,\ v^p,\ w^p,\ \omega_1^p,\ \tilde{\omega}_2^p,\ \omega_3^p,\ \tilde{f}^p,\ \tilde{g}^p\} \tag{37b}$$

$$\mathbf{U}_q^T=\{u^q,\ v^q,\ w^q,\ \omega_1^q,\ \omega_2^q,\ \omega_3^q,\ f^q,\ g^q\} \tag{37c}$$

$$\tilde{\mathbf{U}}_q^T=\{u^q,\ v^q,\ w^q,\ \omega_1^q,\ \tilde{\omega}_2^q,\ \omega_3^q,\ \tilde{f}^q,\ \tilde{g}^q\} \tag{37d}$$

and

$$\mathbf{T}_1=\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1/R & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/R & 1 & \cdot \\ \cdot & \cdot & -1/R & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \tag{38}$$

Based on Eq. (36), equilibrium equation (33) is transformed to

$$(\tilde{\mathbf{K}}_e-\omega^2\tilde{\mathbf{M}}_e)\,\tilde{\mathbf{U}}_e=\tilde{\mathbf{F}}_e \tag{39}$$

where

$$\tilde{\mathbf{U}}_e = \left\langle \begin{matrix} u^p, v^p, w^p, \omega_1^p, \tilde{\omega}_2^p, \omega_3^p, \tilde{f}^p, \tilde{g}^p \\ u^q, v^q, w^q, \omega_1^q, \tilde{\omega}_2^q, \omega_3^q, \tilde{f}^q, \tilde{g}^q \end{matrix} \right\rangle \quad (40a)$$

$$\tilde{\mathbf{F}}_e = \left\langle \begin{matrix} F_1^p, F_2^p, F_3^p, M_1^p, \tilde{M}_2^p, M_3^p, \tilde{M}_\phi^p, \tilde{F}_m^p \\ F_1^q, F_2^q, F_3^q, M_1^q, \tilde{M}_2^q, M_3^q, \tilde{M}_\phi^q, \tilde{F}_m^q \end{matrix} \right\rangle \quad (40b)$$

Matrices and vectors in Eq. (39), respectively, are evaluated as

$$\tilde{\mathbf{K}}_e = \mathbf{T}^T\mathbf{K}_e\mathbf{T},\ \tilde{\mathbf{M}}_e = \mathbf{T}^T\mathbf{M}_e\mathbf{T} \qquad \tilde{\mathbf{U}}_e = \mathbf{T}^T\mathbf{U}_e,\ \tilde{\mathbf{F}}_e = \mathbf{T}^T\mathbf{F}_e \quad (41a\text{-}d)$$

where

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_1 & \cdot \\ \cdot & \mathbf{T}_1 \end{bmatrix} \quad (42)$$

Then the global system of matrix equilibrium equation for the free vibration and elastic analysis of non- symmetric thin-walled curved beam may be obtained using the direct stiffness method.

4. Numerical Examples

In this Section, the free vibration and elastic analysis of curved beam with mono-symmetric and non-symmetric thin-walled cross sections are performed and compared with the solutions obtained from a single reference line (the line of centroid) formulation presented by Kim at al.(2002), solutions by other researchers and ABAQUS's shell elements. Also in subsequent examples, the curved beam is modeled by 20 Hermitian curved beam elements.

4.1 Curved beams with mono-symmetric cross sections

First the simply supported curved beam with mono-symmetric cross section for the x_3 axis which the beam length l is 200 cm and the subtended angle θ_o is 90°, as shown in Fig. 4 is considered. It is well known that the in-plane and out-of-plane behavior of this curved beam is decoupled because the section is mono-symmetric in the plane of beam curvature.

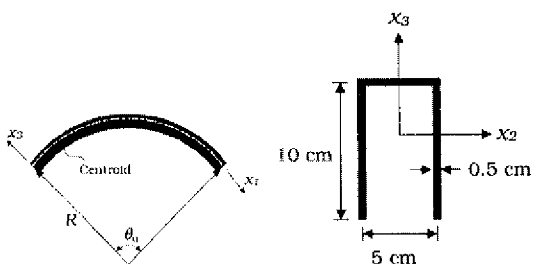
The lowest ten natural frequencies by this study are presented and compared with the solutions based on the centroid formulation which all seven displacements are defined at the centroid in Table 1. And the lateral displacement U_y and the twisting angle θ at the shear center of mid-span of curved beam subjected to torsional moment $M_1 = 10000$ Ncm acting at mid-span by this study are compared with the solutions by the centroid formulation in Table 2. It can be found

Table 1 Natural frequency of simply supported curved beam with x_3 mono-symmetric section, (rad./sec)

Mode	This study	C-formulation
1	1.6579	1.6579
2	33.049	33.049
3	37.792	37.792
4	40.767	40.767
5	44.559	44.559
6	55.998	55.998
7	59.412	59.412
8	84.894	84.894
9	94.375	94.375
10	116.51	116.51

Table 2 Lateral displacement and twisting angle of simply supported curved beam with x_3 mono-symmetric section, (cm, rad.)

Mode	This study	C-formulation
U_y	−1.8911	−1.8911
θ	0.057387	0.057387



(a) Geometry of a curved beam (b) Mono-symmetric cross section for x_3 axis

$E = 2\times10^7$ N/cm², $G = 7692308.$ N/cm², $\rho = 0.077009$ N/cm³, $A = 12.5$ cm², $J = 1.04167$ cm⁴, $e_2 = 0$ cm, $e_3 = 8.61538$ cm, $I_2 = 133.33333$ cm⁴, $I_3 = 67.70833$ cm⁴, $I_{222} = -100$ cm⁵, $I_{233} = -41.66667$ cm⁵, $I_\phi = 641.02564$ cm⁶, $I_{\phi 23} = 641.02564$ cm⁶, $I_{\phi\phi 2} = -486.93294$ cm⁷

(c) Material and sectional properties

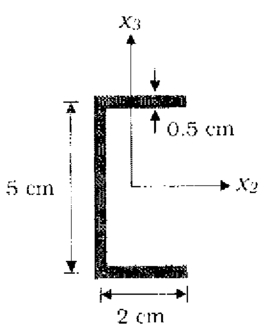
Fig. 4 Simply supported curved beam with mono-symmetric cross section for x_3 axis from Tables 1 and 2 that the natural frequencies and the displacements by this study coincide exactly with the solutions based on the centroid formulation.

Table 3 Natural frequency of simply supported curved beam with x_2 mono-symmetric section, (rad./sec)

Mode	This study	C-formulation
1	3.9294	3.9294
2	27.940	27.940
3	73.536	73.536
4	87.308	87.308
5	91.445	91.445
6	138.77	138.77
7	148.30	148.30
8	158.23	158.23
9	219.67	219.67
10	225.27	225.27

Table 4 Lateral, vertical displacements and twisting angle of simply supported curved beam with x_2 mono-symmetric section, (cm, rad.)

Mode	This study	C-formulation
U_y	-1.8329	-1.8329
U_z	-0.11859	-0.11859
θ	0.11398	0.11398



(a) Mono-symmetric cross section for x_2 axis

$E=2\times10^7$ N/cm^2, $G=7692308.$ N/cm^2,
$\rho=0.077009$ N/cm^3, $A=4.5$ cm^2, $J=0.375$ cm^4
$e_2=-1.15033$ cm, $e_3=0$ cm, $I_2=17.70833$ cm^4,
$I_3=1.77778$ cm^4, $I_{223}=4.62963$ cm^5, $I_{333}=1.23457$ cm^5,
$I_\phi=7.84314$ cm^6, $I_{\phi 23}=-7.84314$ cm^6

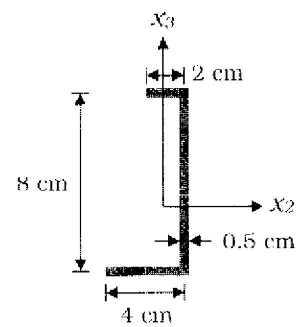
(b) Material and section properties

Fig. 5 Simply supported curved beam with mono-symmetric cross section for x_2 axis

Next, Fig. 5 shows the mono-symmetric cross section for x_2 axis and its material and sectional properties of simply supported curved beam, in which the subtended angle and the beam length are 90° and 100 cm, respectively. In this case, the vibrational and elastic behavior of curved beam is spatially coupled because of the mono-symmetric cross section for x_2 axis. In Tables 3 and 4, the spatially coupled natural frequencies and the displacements at the shear center of loading point of beam subjected to $M_1=10000$ Ncm acting at mid-span are given and compared. The excellent agreement between results based on two formulations is evident.

4.2 Curved beams with non-symmetric cross section

In this example, the non-symmetric curved beams with clamped-free and clamped-clamped boundary conditions at the both ends are considered. Figure 6 shows the configuration of non-symmetric cross section and the material and sectional properties. First, the lowest ten spatially coupled natural frequencies for cantilevered and clamped curved beams for subtended angle 10°



(a) Non-symmetric cross section

$E=294300$ N/cm^2, $G=112815$ N/cm^2,
$\rho=0.077009$ N/cm^3, $A=7$ cm^7, $J=0.58333$ cm^4,
$e_2=1.44846$ cm, $e_3=-2.04461$ cm, $I_2=67.04762$ cm^4,
$I_3=8.42857$ cm^4, $I_{23}=9.14286$ cm^4, $I_{222}=52.24490$ cm^5,
$I_{223}=-20.02721$ cm^5, $I_{233}=-17.41497$ cm^5,
$I_{333}=-13.38776$ cm^5, $I_\phi=42.48664$ cm^6
$I_{\phi 22}=24.48383$ cm^6, $I_{\phi 23}=-42.48664$ cm^6,
$I_{\phi 33}=-10.53165$ cm^6, $I_{\phi\phi 2}=117.44909$ cm^7, $l=200$ cm

(b) Material and section properties

Fig. 6 Cantilevered and clamped curved beams with non-symmetric cross section

Table 5 Natural frequency of cantılevered curved beam wıth non-symmetrıc sectıon, (rad /sec)^2

θ_o		Vıbratıon mode									
		1	2	3	4	5	6	7	8	9	10
10	Thıs study	0 0290	0 2686	0 5963	1 5252	5 1373	7 7437	17 386	20 622	27 159	52 343
	Kım et al (2002)	0 0290	0 2686	0 5963	1 5252	5 1373	7 7437	17 386	20 622	27 159	52 343
	ABAQUS	0 0299	0 2670	0 5887	1 5265	5 0520	7 7433	16 925	20 575	26 645	52 892
90	Thıs study	0 0062	0 2061	0 2901	2 0272	5 2138	7 3645	17.473	32 844	37 949	47 720
	Kım et al (2002)	0 0062	0 2061	0 2901	2 0272	5 2138	7 3645	17 473	32 844	37 949	47 720
	ABAQUS	0 0060	0 2043	0 2779	2 1714	5 0293	7 1815	17 079	32 233	36 624	43 574

Table 6 Natural frequency of clamped curved beam wıth non-symmetrıc sectıon, (rad /sec)^2

θ_o		Vıbratıon mode									
		1	2	3	4	5	6	7	8	9	10
10	Thıs study	0 9488	4 4120	6 3262	17 731	18 778	21 295	49 633	59 534	99 774	119 58
	Kım et al (2002)	0 9488	4 4120	6 3262	17 731	18 778	21 295	49 633	59 534	99 774	119 58
	ABAQUS	0 9679	4 3543	6 4045	16 946	18 565	21 369	50 231	58 585	100 44	105 01
90	Thıs study	0 7223	3 9916	13 570	31 829	35 223	41 852	71 047	80 658	138 20	148 88
	Kım et al (2002)	0 7223	3 9916	13 570	31 829	35 223	41 852	71 047	80 658	138 20	148 88
	ABAQUS	0 7020	3 9088	13 388	30 838	34 855	37 792	69 831	78 659	115 15	140 53

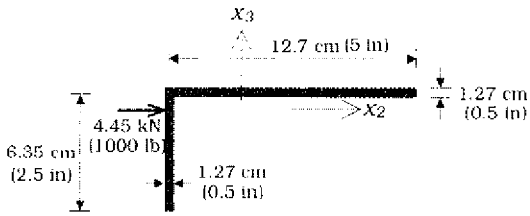
Table 7 Lateral, vertıcal dısplacements and twıstıng angle of clamped curved beam with non-symmetrıc sectıon, (cm, rad)

Mode	Thıs study	C-formulatıon
U_y	−1 4185	−1 4185
U_z	0 12054	0 12054
θ	0 16893	0 16893

and 90° wıth keepıng the total length of beam constant by thıs study are presented ın Tables 5 and 6, respectıvely For comparison, the prevıous solutıons based on the centroıd formulatıon (Kım at al, 2002) and the results obtaıned from 300 nıne-noded shell elements (S9R5) of ABAQUS whıch ıs the commercıal fınıte element analysıs program are gıven From Tables 5 and 6, ıt can be observed that the centroıd-shear center formulatıon proposed by thıs study for the vıbratıon analysıs of curved beam wıth non- symmetrıc cross section is accomplished Also results by thıs study are ın a good agreement wıth those by ABAQUS's shell elements Addıtıonally the lateral U_y, vertıcal U_z dısplacements and the twıstıng angle θ at the shear center of mıd-span for clamped curved beam subjected to a torsıonal moment 1000 Ncm at the mıd-span are presented together wıth the results based on the centroıd formulatıon ın Table 7, where exact agreement ıs observed for the spatıally coupled elastıc analysıs of curved beam wıth non-symmetrıc cross sectıon

4.3 Curved beam with L-shaped cross section

We concern the free vıbratıon and elastıc analysıs of the L-shaped curved beam as shown ın Fıg 7 The purpose of thıs example ıs to show the usefulness of the proposed curved beam theory wıth non-symmetrıc sectıon neglectıng warpıng deformatıon and to verıfy how ıt predıcts well the behavıor of structure by comparıng the present solutıons wıth those by ABAQUS's shell elements and the prevıous researches The curved beam ıs the clamped at the both ends and subjected to



(a) Non-symmetric L-shaped cross section

E=20684.28 kN/cm^2, G=7955.49 kN/cm^2,
ρ=0.077009 N/cm^3, A=24.1935 cm^2,
J=13.00723 cm^4, e_2=−4.23333 cm,
e_3=1.05833 cm, I_2=81.29520 cm^4, I_3=433.57440 cm^4,
I_{23}=108.39360 cm^4, I_{222}=−229.43312 cm^5,
I_{223}=−229.43312 cm^5, R=914.4 cm, l=609.6 cm

(b) Material and section properties

Fig. 7 Clamped curved girder with non-symmetric L-shaped section

Table 8 Natural frequency of clamped curved beam with L-shaped section, (rad./sec)^2

Mode	This study	C-formulation	ABAQUS
1	5.6246	5.6246	5.5925
2	6.0305	6.0305	6.1635
3	10.792	10.792	11.001
4	17.427	17.427	17.224
5	19.116	19.116	19.461
6	23.800	23.800	23.917
7	28.498	28.498	28.332
8	30.329	30.329	30.585
9	34.996	34.996	34.712
10	36.862	36.862	36.129

out-of-plane lateral force 4.45 kN (1000lb) acting at the mid-span. In Table 8, the lowest ten spatially coupled natural frequencies by this study using Eqs. (25) and (26) are reported together with those by previous research using Eqs. (27) and (29), which several sectional properties may be needed additionally for analysis and with those obtained from 240 shell elements of ABAQUS. From Table 8, it can be found that present solutions coincide exactly with those by previous research based on the centroid formulation and for comparing with results by ABAQUS, excellent agreement is observed with less than 2.2% as maximum of difference. It should be noted that the present curved beam theory with non-symmetric cross section which the warping function is zero at the shear center eliminates the sectional properties of structures for the dynamic analysis of curved structures.

Next, the lateral displacement U_y at the corner of the L-shaped cross section along the curved beam subjected to out-of-plane lateral force is evaluated and plotted in Fig. 8. By considering the symmetry, 10 curved beam elements are used. For comparison, the results using Eq. (27) and 8 HMC2 curved beam elements by Gendy and Saleeb (1992) based on the centroid formulation and the solutions using 24 quadrilateral shell elements developed by Saleeb et al. (1990) are presented. Investigation of Fig. 8 reveals that present solutions using Eq. (25) are in a good agreement with those obtained from HMC2 elements and shell elements.

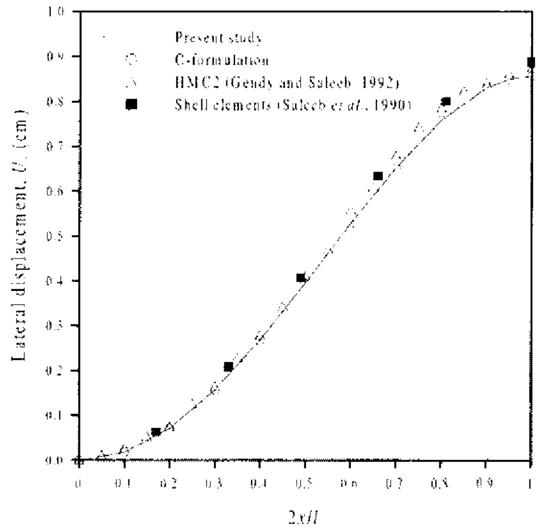
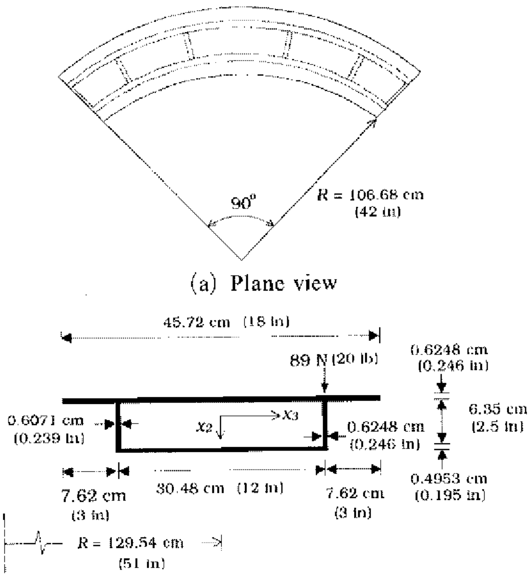


Fig. 8 Lateral displacement at the shear center of L-shaped girder

4.4 Curved box girder with non-symmetric cross section

In our final example, the non-symmetric curved box girder as shown in Fig. 9 is considered. The girder is simply supported at the two ends and is subjected to an eccentric lateral force 89 N (20lb) at the exterior web of mid-span. Because the material properties of plexiglass are time dependent, a series of preliminary tension and bending tests were performed on specimens cut



(a) Plane view

(b) Non-symmetric box section

Fig. 9 Simply supported non-symmetric curved box girder

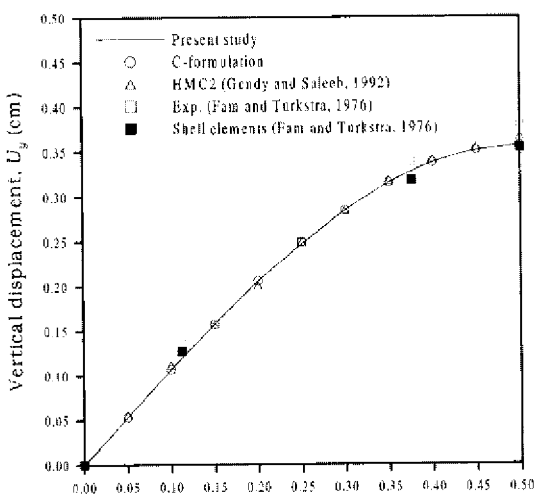


Fig. 10 Vertical displacement along the external web of a curved box girder

from the same sheet as the model sections. As a result of test, the material properties are taken as E=275.97 kN/cm^2 (400 ksi) and poisson's ratio v=0.36. To prevent the distortion of cross section of box girder, two end diaphragms and four intermediate diaphragms at angles of 15°, 35°, 55°, and 75° from the lines of support are installed. By considering the symmetry, one half of span is modeled by 10 elements. Out-of-plane lateral displacement of the top flange at the location of the exterior web of midspan is shown in Fig. 10. For comparison, the results by the centroid formulation, 10 HMC2 elements, experimental results and FE solutions using shell elements by Fam and Turkstra (1976) are presented. From Fig. 10, it can be found that present results are in a good agreement with the comparisons reported. Consequently the analysis neglecting the warping deformation results in an excellent fit to the behavior of curved box girders.

5. Conclusions

A centroid-shear center formulation for the spatially coupled free vibration and elastic analysis of thin-walled curved beams with non-symmetric open and closed cross sections is proposed. This theory overcomes the drawback of previous curved beam theory based on the centroid formulation which should account for sectional properties additionally for curved beams with L- or T-shaped sections. In numerical examples, FE solutions using Hermitian curved beam elements by this study are compared with those obtained from the centroid formulation and the results by available references and ABAQUS's shell elements. Consequently, the following conclusions may be drawn.

(1) The vibration and elastic theories of the thin-walled curved beam neglecting the restrained warping torsion at the shear center may be easily derived from the thin-walled curved beam theory based on the centroid- shear center formulation by putting the sectional properties associated with warping to zero.

(2) For vibration and elastic analysis of curved beams with mono-symmetric and non-symmetric cross sections, the solutions by this study coincide exactly with those from the centroid formulation.

(3) For curved beam with L-shaped cross section, the natural frequencies and the displacements obtained from this curved beam elements

are in excellent agreement with those from curved beam elements including the warping and ABAQUS's shell elements Resultantly it is believed that this study eliminates total sectional properties of structures for the dynamic and elastic analysis of curved structures

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